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Some Thoughts on the GCF in the Teaching and Learning of School Mathematics

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In this submission I like to do two things. First, I like to comment on claims Duncan McDougall made in *The Math Journal* (Vol. 33(2), pp. 21-24) about the role the concept of the greatest common factor (g.c.f.) plays in the justification and limitations of certain mathematical procedures. Second, I like to discuss a general issue of fundamental concern for the teaching and learning of school mathematics that I see exemplified in the concept of the g.c.f. and McDougall's discussion of the role of the g.c.f. in the teaching and learning of school mathematics.

In his article, McDougall starts out with five questions concerning the justification and limitations of particular mathematical procedures. He claims that the answer to all the questions is grounded in the fact that the g.c.f. for the respective pair of numbers involved in the questions is 1. I like to have a closer look at the last three of those questions. The third of McDougall's questions asks why we cannot combine $\sqrt{3}+\sqrt{7}$ (in his original question he uses "$\sqrt{5}+\sqrt{7}$"), and he answers "We cannot combine $\sqrt{3}+\sqrt{7}$ because g.c.f. (3,7) = 1" (p. 21). However, that g.c.f. (3,7) = 1 *cannot* be the reason, why $\sqrt{3}+\sqrt{7}$ cannot be simplified into a single term. The first interesting point is the ambiguity of a 'because' statement in a mathematical explanation. In order to count as an explanation, the fact that $\sqrt{3}+\sqrt{7}$ cannot be simplified into a single term and the fact that g.c.f. (3,7)

= 1 have to relate to each other in a systematic and generalizable way. The ‘because’ explanation, however, allows for *two different* readings of such a systematic and general relationship between the g.c.f. and the limit of simplifying radicals: (1) whenever g.c.f. $(a,b) \neq 1$, then $\sqrt{a} + \sqrt{b}$ can be combined; (2) whenever g.c.f. $(a,b) = 1$, then $\sqrt{a} + \sqrt{b}$ cannot be combined. Both statements are not logically equivalent. The reason why McDougall’s explanation for why $\sqrt{3} + \sqrt{7}$ cannot be combined is not correct is because both readings of his claim have counter examples. (1) For $a = 2$ and $b = 6$, the first general statement that could justify McDougall’s claim is not true: g.c.f. $(2,6) \neq 1$, but $\sqrt{2} + \sqrt{6}$ *cannot* be simplified into a single term. (2) For $a = 4$ and $b = 9$, the second general statement that could justify McDougall’s claim is not true: g.c.f. $(4,9) = 1$, but $\sqrt{4} + \sqrt{9}$ *can* be simplified into a single term. So, although it is true that $\sqrt{3} + \sqrt{7}$ cannot be simplified into a single term, and it is true that g.c.f. $(3,7) = 1$, the former *is not* because of the latter as the counter example for each of the two readings of McDougall’s claim show.

The fourth question, McDougall raises, asks: “Why is the lowest common multiple of sin x and cos x their product?”, and he argues that the same reason as in the first three questions holds true here as well: g.c.f. (sin x, cos x) = 1. The problem with this argument is that it does not make sense (in a semantic sense) to talk about the g.c.f. of two functions. The problem already starts when one tries to talk about ‘factors of functions’ (which one needs to do before one can talk about *common* factors of two functions and then talk about the *greatest* of those common factors). Do functions have factors? Such a notion would be very problematic for a similar reason why it is problematic to talk about factors of rational numbers. Every rational number is a factor of every rational number, which makes the idea of a factor of rational numbers rather useless: If a and b are rational numbers, then a can be written as $a = b \times c$, where c is the quotient of a over b with b being a factor of a. Similarly with functions: If $f(x)$ and $g(x)$ are two functions over the same domain and $g(x) \neq 0$ for all elements in the domain, then $f(x) = g(x) \times h(x)$, where $h(x) = \frac{f(x)}{g(x)}$. But assuming that one, nevertheless, wants to talk about factors of functions, then g.c.f. (sin x, cos x) is *not* equal to 1, because both functions have common factors: Since $\sin x = \cos x \times \tan x$ and $\cos x = \sin x \times \cot x$, sin x and cos x is each a common factor of sin *x* and cos *x*. There is also another problem involved here, namely that there is generally no order structure defined for functions as it is for whole numbers (integers). To illustrate the problem using the example of the trigonometric functions: If sin *x* and cos *x* are both common factors of sin *x* and cos *x*, which of the two factors is the *greater* one? There is no order defined for trigonometric functions, so it is meaningless (in a semantic sense) to talk about the *greatest* common factor of given trigonometric functions.

In his fifth and last example, McDougall asks: "Why can we 'cross-multiply' to get equivalent terms for $\frac{1}{x+h} - \frac{1}{x}$?" (p. 21). Here, he also refers to g.c.f. $(x + h, x) = 1$ as the reason for the procedure to work. First, it is unusual to talk about 'cross-multiplying' for the procedure he seems to refer to. The term is usually used for the short-cut to get an equivalent equation to an equation with two rational expressions on either side. For instance, one can find an equivalent equation to $\frac{1}{x+h} = \frac{1}{x}$ by 'multiplying numerators and denominators across the equal sign to get to the equivalent equation $x \times 1 = (x + h) \times 1$ (ignoring the different restrictions on domains). This procedure involves a true 'cross multiplying'. The procedure McDougall seems to refer to would require more than such a 'cross-multiplying' (across the operation sign), because there is in addition also an 'across-multiplying' involved in order to get to the equivalent term $\frac{1 \times x - (x+h) \times 1}{(x+h) \times x}$. Turning now to the claim that g.c.f. $(x + h, x) = 1$ is the reason for the procedure to result in an equivalent term: The factors of the denominators in the two terms making up the expression the procedure is applied to have *no* influence on the procedure to produce an equivalent term to the original expression. The procedure *always* leads to an equivalent term to the original expression. For the first reading of the 'because' explanation (for the two readings see above), the following provides a counter example: g.c.f. $(x^2, x) \neq 1$, but the 'cross-multiplying' procedure applied to $\frac{1}{x^2} - \frac{1}{x}$ leads to the equivalent term $\frac{1 \times x - x^2 \times 1}{x^2 \times x}$. Using the second reading of the 'because' expression (whenever the g.c.f of the two denominators is equal to one, the procedure will result in an equivalent expression) cannot serve as a reason either, because the factors of the two denominators have nothing to do with the procedure to produce an equivalent term.

I now like to turn to a general issue that I see raised by McDougall's discussion of the role the g.c.f. in the teaching and learning of school mathematics. The general issue is that school mathematics is impacted – generally to the detriment of developing understanding of mathematical concepts and ideas in many students – by mathematicians' desire and practice of conciseness in the representation of mathematical ideas – a conciseness that mathematicians often connect with elegance and aesthetics. The notion of the g.c.f. exemplifies this practice as well as the challenges that come with this practice for school students' engagement with mathematical ideas. I will use McDougall's first g.c.f. example – which I did not discuss above – to illustrate this practice of conciseness as well as its impact on the teaching and learning on school mathematics.

In his first g.c.f. example, McDougall asks why we cannot reduce $\frac{3}{7}$: "We cannot reduce $\frac{3}{7}$ because g.c.f. (3,7) = 1." (p. 21) It is an elegant, some might even suggest 'aesthetically pleasing' way of expressing a reason why a fraction cannot be reduced further. It is an elegant answer, because a *simple* equation expresses *symbolically* that the numerator and the denominator have no non-trivial factor in common by which both numbers can be reduced (evenly divided). From a learning and teaching perspective, however, it is important to note that this elegance brings with it a more complex mathematical language to be understood by students, here through the introduction of the complex term 'g.c.f.'. The 'trade-off' that generally comes with the introduction of a complex term in order to be able to express more complex situations in an easier way is quite nicely illustrated in McDougall's article. To illustrate the role of the g.c.f. in the process of reducing a fraction like $\frac{57}{95}$, McDougall writes: "Many students would first wonder whether it was possible, and even if it were, how then to proceed. The quickest method is to calculate g.c.f. (57,95) using Euclid's Algorithm." (p. 22). Euclid's Algorithm, however, is here not the fastest method to find g.c.f (57,95) – rather using divisibility rules and a maximum of two division operations following that. Furthermore, to wonder about g.c.f. (57,95) is not even required in reducing the fraction. The understanding of the big mathematical idea of equivalent fractions includes the understanding that a fraction can be reduced if numerator and denominator have a common whole-number factor other than 1. Reducing fractions, then, is simply a matter of finding common factors – no g.c.f. required. We do not need the concept of a g.c.f. in order to understand reducing fractions and, thus, non-reducible fractions. Considering that the Euclidean Algorithm – compared to the notion of equivalent fractions – will have to stay a mathematical procedure that students of any school grades will have to apply blindly without a good chance of understanding why it works, the teaching and learning value of the g.c.f. for explaining non-reducible fractions is questionable.

Another example for this type of conciseness in the mathematical sciences that has a problematic impact on the teaching and learning of school mathematics is the use of absolute value to characterize distance sets on a number line. For instance, the set of all real numbers that are more than two units away from six can be quite concisely written as $|x-6| \geq 2$. However, it can also be written as the set of numbers x for which $x < 4 \;\; OR \;\; x > 8$. This expression is not as concise as the absolute value one because the values of '6' and '2', which are listed in the problem, do not show up directly, but the absolute value – which introduces a more complex language into school mathematics – is in this context not required.

The same general concern about the impact of the principle of conciseness in the mathematical science on the meaningful teaching and learning of school mathematics applies in cases where the conciseness manifests itself in the creation or acceptance of ambiguities, which make the understanding of school mathematics unnecessarily complicated for students. Here are two examples. The first example is the use of the same symbol for the subtraction operation and for marking a number as negative. For the proficient speakers of the mathematical language, this ambiguity is no problem because the context makes the meaning clear, but it can provide the learner of school mathematics with frustrating experiences. The second example is the use of the same expressions for equations and inequalities with one variable and special equations and inequalities with two variables. For instance, the expressions '$y = 1$' and '$x \leq 3$' can stand for equations and inequalities, respectively, with one or two variables, and the solution sets are quite different accordingly. Difficulties arising from the tendency in the mathematical science to use a principle of conciseness, which makes the mathematical language more complex or more ambiguous to understand, can be avoided if the principle of conciseness is left to those who feel at home in the mathematical sciences (and that might involve some school students) and the understanding of big mathematical ideas and important concepts is not made more complicated than necessary. Why should we, who are interested in students' learning of big mathematical ideas and important mathematical concepts, not modify the language of school mathematics in a way that helps students understand these ideas and concepts better, rather than being concerned about keeping the language of the mathematical science pure but unnecessarily complex? McDougall describes the g.c.f. as 'the silent partner' (in understanding mathematical operations and processes) and concludes his article with "This partner need not be silent anymore." (p. 24) Maybe more for the sake of provoking thinking about a taking-for-granted tradition of adopting the language of the mathematical science in school mathematics, I like to suggest that teachers of school mathematics have good reasons to keep this partner silent for the benefit of the learning of what really matters in school mathematics.